MATH 150 **Exam 3 Notes** **Fall**

1 Derivatives of Polynomials and Exponentials

Suppose that $f(x) = c$, that is, f is a constant function. Then:

$$f'(x) \quad = \quad \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \quad = \quad \lim_{h\to 0} \frac{c-c}{h} \quad = \quad 0$$

Therefore,

$$\frac{d}{dx}(c) = 0$$

What about the power functions, $f(x) = x^n$?

Observe that

$$f'(x) \quad = \quad \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \quad = \quad \lim_{h\to 0} \frac{(x+h)^n - x^n}{h}$$

Using the BINOMIAL THEOREM we get:

$$(x+h)^n \quad = \quad * \ \left[x^n + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \ldots + nxh^{n-1} + h^n\right]$$

So now we have:

$$f'(x) \quad = \quad \lim_{h\to 0} \frac{[*] - x^n}{h} \quad =$$

$$\lim_{h\to 0} \left[nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}h + \ldots + nxh^{n-2} + h^{n-1}\right] \quad = \quad nx^{n-1}$$

Which gives us the

Power Rule

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

ex 1 Find $f'(x)$ for the following:

1. $f(x) = x^5$
2. $f(x) = x^{500}$
3. $f(x) = \sqrt[3]{x^4}$
4. $f(x) = \frac{1}{x^2}$

Using the Limit Laws, it is easy to show that $\frac{d}{dx}[c \cdot f(x)] = c \cdot \frac{d}{dx} f(x)$ for any constant c

ex 2 Find y' for the following:

1. $f(x) = 3x^7$
2. $f(x) = -2x$

Again, using those same Limit Laws, if f and g are both differentiable then

$$\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}(f(x)) \pm \frac{d}{dx}(g(x))$$

ex 3 Find the points on the curve $y = x^4 - 6x^2 + 4$ where the tangent line is horizontal

That's the same as asking when does $y' = 0$.

$$y' = 4x^3 - 12x \;=\; 0 \iff 4x(x^2-3) \;=\; 0 \iff x = 0, \sqrt{3} \; or \; -\sqrt{3}$$

How about EXPONENTIAL FUNCTIONS ?

If $f(x) = a^x \implies$

$$f'(x) \;=\; \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \;=\; \lim_{h\to 0} \frac{a^{x+h}-a^x}{h} \;=$$

$$\lim_{h\to 0} \frac{a^x(a^h-1)}{h} \;=\; a^x \cdot \lim_{h\to 0} \frac{a^h-1}{h}$$

What is

$$\lim_{h\to 0} \frac{a^h-1}{h} \; ? \qquad \implies f'(0) \; !$$

THE RATE OF CHANGE OF AN EXPONENTIAL FUNCTION IS PROPORTIONAL TO THE FUNCTION ITSELF !

$$for \;\; a = 2 \qquad f'(0) \;=\; \lim_{h\to 0} \frac{2^h-1}{h} \;\approx\; .69$$

$$for \;\; a = 3 \qquad f'(0) \;=\; \lim_{h\to 0} \frac{3^h-1}{h} \;\approx\; 1.1$$

In fact

$$\frac{d}{dx}(a^x) \;=\; a^x \cdot ln\; a$$

So the question is .... does there exist a number between 2 and 3 such that $f'(0) = 1$?

That is why we love e

$$\frac{d}{dx}(e^x) = e^x$$

ex 4 Find y' if $y = 2e^x - x^2 + 5$

Worksheet for Section 1

1. Find the derivatives of the following functions:

   (a)
   $$f(x) = \sqrt{30}$$

   (b)
   $$f(x) = 5e^x + 3$$

   (c)
   $$f(x) = \frac{\sqrt{10}}{x^7}$$

   (d)
   $$f(x) = \sqrt{x} - \frac{1}{\sqrt{x}}$$

2. Find the equation of the tangent line to $y = x^4 + 2e^x$ at $(0, 2)$

Homework for Section 1

1. Differentiate the following functions:

   (a) $f(x) = 186$

   (b) $f(x) = 2 - \frac{2}{3}x$

   (c) $f(x) = x^3 - 5x + 9$

   (d) $f(x) = \frac{1}{3}(x^5 + 7)$

   (e) $f(x) = x^{-2/5}$

   (f) $f(x) = \frac{4}{3}\pi x^3$

   (g) $f(x) = -\frac{12}{x^6}$

   (h) $f(x) = \sqrt{x} - 2e^x$

   (i) $f(x) = ax^2 + bx + c$

   (j) $f(x) = \frac{x^2+4x+2}{\sqrt{x}}$

   (k) $f(x) = e^{x+1} + 1$

2. Find an equation of the tangent line to $f(x) = 3x^2 - x^3$ at the point$(1, 2)$.

3. Find the first and second derivative of the following function:

   (a) $f(x) = x^4 - 3x^3 + 8x$

4. For what values of x does the graph $f(x) = x^3 + 3x^2 + x + 3$ have a horizontal tangent?

2 The Product and Quotient Rules

Consider the following functions:

$$f(x) = x \text{ and } g(x) = x^2$$

What do you think $(fg)'$ would be?

In other words, is the derivative of a product of two functions the product of the derivatives?

$$f(x) = x, \;\; g(x) = x^2 \;\; and \;\; f(x)g(x) = x^3$$

However

$$f'(x) = 1 \;\; and \;\; g'(x) = 2x \;\; so \;\; f'(x) \cdot g'(x) = 2x \;\; \neq \;\; (f(x)g(x))' = 3x^2$$

So we have ...

The Product Rule

If f and g are both differentiable then

$$\frac{d}{dx}[f(x)g(x)] \;\; = \;\; f(x) \cdot \frac{d}{dx}(g(x)) + g(x) \cdot \frac{d}{dx}(f(x))$$

Or

$$(fg)' \;\; = \;\; f \cdot g' + g \cdot f'$$

ex 5 Find $f'(x)$ if $f(x) = x^2 e^x$

Let $f(x) = x^2$ and $g(x) = e^x$ and we get

$$\frac{d}{dx}\left[f(x)g(x)\right] \;=\; x^2 \cdot \frac{d}{dx}e^x + e^x \cdot \frac{d}{dx}x^2 \;=\; x^2e^x + e^x(2x)$$

ex 6 Find $f'(x)$ if $f(x) = \sqrt{x}(1-x)$

Let $f(x) = \sqrt{x}$ and $g(x) = (1-x)$ and we get

$$\frac{d}{dx}\left[f(x)g(x)\right] \;=\; \sqrt{x}\cdot\frac{d}{dx}(1-x)+(1-x)\cdot\frac{d}{dx}\sqrt{x} \;=\; -\sqrt{x}+(1-x)\frac{1}{2\sqrt{x}}$$

ex 7 If $f(x) = \sqrt{x}g(x)$ and $g(4) = 2 \; and \; g'(4) = 3$ find $f'(4)$

$$\begin{aligned} f'(x) \;&=\; \sqrt{x}\cdot\frac{d}{dx}g(x) + g(x)\cdot\frac{d}{dx}\sqrt{x} \;=\; \sqrt{x}g'(x) + g(x)\frac{1}{2\sqrt{x}} \\ \implies \; f'(4) \;&=\; \sqrt{4}g'(4) + g(4)\frac{1}{2\sqrt{4}} \;=\; 6 + \frac{1}{2} \end{aligned}$$

What about the quotient of two functions?

If we let

$$F(x) = \frac{f(x)}{g(x)} \implies F(x)g(x) = f(x) \implies f'(x) = F(x)g'(x)+g(x)F'(x)$$

$$so \; F'(x)g(x) \;=\; f'(x)-F(x)g'(x) \implies F'(x)g(x) \;=\; f'(x)-\frac{f(x)}{g(x)}g'(x)$$

$$\implies \; F'(x) \;=\; \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{[g(x)]^2}$$

Which gives us ...

The Quotient Rule

$$F'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

ex 8 Find y' if

$$y = \frac{2x^3 + x^2 - 3}{x^2 - 6}$$

So if $f(x) = 2x^3 + x^2 - 3$ and $g(x) = x^2 - 6$ we have

$$y' = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2} = \frac{(x^2-6)(6x^2+2x) - (2x^3+x^2-3)(2x)}{(x^2-6)^2}$$

You do NOT need to simplify this

ex 9 Find the equation of the tangent line to the following curve at $(1, e/2)$

$$y = \frac{e^x}{1+x^2}$$

So if $f(x) = e^x$ and $g(x) = 1 + x^2$ we have

$$y' = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2} = \frac{(1+x^2)(e^x) - (e^x)(2x)}{(1+x^2)^2}$$

So the slope when $x = 1$ is

$$m = \frac{2e - 2e}{4} = 0 \implies \textit{the equation is } y = \frac{e}{2}$$

ex 10 Find $F'(x)$ if

$$F(x) = \frac{3x^2 + 2\sqrt{x}}{x}$$

So

$$F(x) = \frac{3x^2}{x} + \frac{2\sqrt{x}}{x} = 3x + 2x^{-1/2} \implies F'(x) = 3 - x^{-3/2}$$

The moral here is sometimes it's easier to simplify.

Worksheet for Section 2

1. Find the derivatives of the following functions:

   (a)
   $$f(x) = \frac{x - 3x\sqrt{x}}{\sqrt{x}}$$

   (b)
   $$g(x) = \sqrt{x}e^x$$

   (c)
   $$y = \frac{\sqrt{x} - 1}{\sqrt{x} + 1}$$

2. If $f(3) = 4$, $g(3) = 2$, $f'(3) = -6$ and $g'(3) = 5$ find $(fg)'(3)$

Homework for Section 2

1. Differentiate:

   (a)
   $$f(x) = (x^3 + 2x)e^x$$

   (b)
   $$g(x) = \sqrt{x}e^x$$

   (c)
   $$y = \frac{e^x}{x^2}$$

   (d)
   $$y = \frac{e^x}{1+x}$$

   (e)
   $$g(x) = \frac{3x+1}{2x+1}$$

   (f)
   $$f(t) = \frac{2t}{4+t^2}$$

   (g)
   $$y = (2x^3 + 3)(x^4 - 2x)$$

   (h)
   $$y = \frac{t}{(t-1)^2}$$

2. Use the Product Rule twice to prove that if f, g *and* h are differentiable then $(fgh)' = f'gh + fg'h + fgh'$

3 Derivatives of Trig Functions

Recall: We will ALWAYS use radian measure in calculus.

If you graph $y = sin\ x$ and on the same set of axes graph y', what does the graph look like?

If you have graphed carefully enough it y' should closely resemble $cos\ x$

Let's confirm this by going to the definition

$$\begin{aligned} y' &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{sin(x+h)-sin\ x}{h} = \\ &\lim_{h\to 0} \frac{sin\ x\ cos\ h + cos\ x\ sin\ h - sin\ x}{h} \\ &(using\ trig\ identities) \\ &= \lim_{h\to 0} \left[\frac{sin\ x\ cos\ h - sin\ x}{h} + \frac{cos\ x\ sin\ h}{h}\right] = \dots \\ &= \lim_{h\to 0} sin\ x \cdot \lim_{h\to 0} \frac{cos\ h - 1}{h} + \lim_{h\to 0} cos\ x \cdot \lim_{h\to 0} \frac{sin\ h}{h} \\ &= sin\ x \cdot \lim_{h\to 0} \frac{cos\ h - 1}{h} + cos\ x \cdot \lim_{h\to 0} \frac{sin\ h}{h} \end{aligned}$$

So we need to evaluate

1. $\lim_{h\to 0} \frac{sin\ h}{h}$
2. $\lim_{h\to 0} \frac{cos\ h-1}{h}$

The first one is a geometric argument which I can show you time permitting.

The second we can do algebraically.

so

$$\lim_{h\to 0} \frac{cos\ h - 1}{h} \ = \ \lim_{h\to 0} \left[\frac{cos\ h - 1}{h} \cdot \frac{cos\ h + 1}{cos\ h + 1}\right]$$

$$= \lim_{h\to 0} \frac{cos^2\ h - 1}{h(cos\ h + 1)} = \lim_{h\to 0} \frac{-sin^2\ h}{h(cos\ h + 1)} = -\lim_{h\to 0} \frac{sin\ h}{h} \cdot \frac{sin\ h}{cos\ h + 1}$$

$$= \ -\lim_{h\to 0} \frac{sin\ h}{h} \cdot \lim_{h\to 0} \frac{sin\ h}{cos\ h + 1} \ = \ -1\left(\frac{0}{1+1}\right) \ = \ 0$$

Clearly we now know that

$$\lim_{h\to 0} \frac{sin\ h}{h} \ = \ 1$$

Putting everything together we get ...

$$f'(x) \ = \ \lim_{h\to 0}\ sin\ x \cdot \lim_{h\to 0} \frac{cos\ h - 1}{h} + \lim_{h\to 0}\ cos\ x \cdot \lim_{h\to 0} \frac{sin\ h}{h}$$

$$= \ (sin\ x)(0) + (cos\ x)(1) \ = \ cos\ x$$

Thus

$$\frac{d}{dx}(sin\ x) \ = \ cos\ x$$

Also

$$\frac{d}{dx}(cos\ x) \ = \ -\ sin\ x$$

We could find the derivative of $tan\ x$ using what? The quotient rule.

You will need to know the following:

Derivatives of Trigonometric Functions

$$\frac{d}{dx}(sin\ x) = cos\ x \qquad \frac{d}{dx}(csc\ x) = -\ csc\ x\ cot\ x$$

$$\frac{d}{dx}(cos\ x) = -\ sin\ x \qquad \frac{d}{dx}(sec\ x) = sec\ x\ tan\ x$$

$$\frac{d}{dx}(tan\ x) = sec^2\ x \qquad \frac{d}{dx}(cot\ x) = -\ csc^2\ x$$

ex 11 If $y = x^3\ sin\ x$, find y'

ex 12 Find

$$\lim_{x \to 0} \frac{sin\ 7x}{4x}$$

Note that

$$\lim_{x \to 0} \frac{sin\ 7x}{4x} \cdot \frac{7}{7} = \lim_{x \to 0} \frac{sin\ 7x}{7x} \cdot \frac{7}{4} = \frac{7}{4} \cdot \lim_{x \to 0} \frac{sin\ 7x}{7x} = \frac{7}{4} \cdot 1 = \frac{7}{4}$$

Worksheet for Section 3

1. Find the derivative of $f(x) = x \; sin \; x$
2. Find an equation of the tangent line to $y = e^x \; cos \; x$ at $(0, 1)$
3. Evaluate

$$\lim_{t \to 0} \frac{sin^2 \, 3t}{t^2}$$

Homework for Section 3

1. Differentiate:

   (a) $f(x) = 3x^2 - 2\ cos\ x$

   (b) $f(x) = sin\ x + \frac{1}{2} cot\ x$

   (c) $g(x) = x^3 cos\ x$

   (d) $f(\theta) = csc\ \theta + e^{\theta} cot\ \theta$

   (e)
   $$y = \frac{x}{2 - tan\ x}$$

   (f)
   $$y = \frac{sec\ \theta}{1 + sec\ \theta}$$

   (g)
   $$y = \frac{sin\ x}{x^2}$$

2. Find an equation of the tangent line to $y = x + cos\ x$ at the point$(0, 1)$

3. Find the following limit:

$$\lim_{x \to 0} \frac{sin\ 3x}{x}$$

4 The Chain Rule

Previously, how did we differentiate $\sqrt{x}$? Rewrite as $x^{1/2}$ and use the power rule ...

But how would we differentiate $\sqrt{x^2+1}$? This is slightly different in that we do not have x to a power but *stuff* to a power.

Note that $\sqrt{x^2+1}$ is a composite function. The chain rule shows us how to deal with these.

THE CHAIN RULE

If f and g are differentiable and $F = f \circ g = f(g(x)) \implies$

$$F'(x) = f'(g(x)) \cdot g'(x)$$

OR

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

ex 13 So, find $F'(x)$ if $F(x) = \sqrt{x^2+1}$

$$F(x) = f(g(x)) \;\; with \;\; g(x) = x^2 + 1 \;\; and \;\; f(u) = \sqrt{u}$$

$$so \;\; f'(u) = \frac{1}{2\sqrt{u}} \;\; and \;\; g'(x) = 2x \implies F'(x) = \frac{1}{2\sqrt{x^2+1}} \cdot 2x = \frac{x}{\sqrt{x^2+1}}$$

There is another way to think about this

The Chain Rule is simply a generalization of all of the previous rules we have covered. In other words,

The Power Rule says

$$\frac{d}{dx}(x^n) \;=\; n \cdot x^{n-1}$$

The Chain rule allows a generalization. That is,

$$\frac{d}{dx}([stuff\ in\ x]^n) \;=\; n \cdot [stuff\ in\ x]^{n-1} \cdot [stuff\ in\ x]'$$

This still applies to x^n, its just that $x' = 1$
We will discuss more generalizations in class

ex 14 If $y = (2x+3)^3(x^2 - x + 4)^2$ find y'

Here we need the product rule. As you are doing the product rule you may have to generalize some of the other rules.

Try not to think of the chain rule as a separate rule but something that is always in effect when you have a composite function

$$If\ \ f(x) = (2x+3)^3 \ \ and\ \ g(x) = (x^2 - x + 4)^2 \ \ then$$

$$y' \;=\; fg' + gf' \;=\; (2x+3)^3 \cdot 2(x^2-x+4)^1(2x-1) + (x^2-x+4)^2 \cdot 3(2x+3)^2(2)$$

There isn't much to simplify here so this is fine as is.

ex 15 Differentiate $y = e^{sin\ x}$

Here we do not have e^x but $e^{(function\ of\ x)}$ so we will need to use the chain rule.

Recall that

$$\frac{d}{dx}e^x \;=\; e^x \quad\Longrightarrow\quad \frac{d}{dx}e^{f(x)} \;=\; e^{f(x)} \cdot f'(x)$$

So

$$\frac{d}{dx}e^{sin\ x} \;=\; e^{sin\ x} \cdot \frac{d}{dx}(sin\ x) \;=\; e^{sin\ x} cos\ x$$

Worksheet for Section 4

1. Find the derivative of the following functions:

   (a) $F(x) = \sqrt{1 + 2x + x^3}$

   (b) $g(t) = \frac{1}{(t^4+1)^3}$

   (c) $y = cos(x^3)$

   (d) $y = e^{-\pi x}$

   (e) $y = xe^{-x^2}$

   (f) $y = e^{xcosx}$

   (g) $F(z) = \sqrt{\frac{z-1}{z+1}}$

Homework for Section 4

1. Differentiate:

   (a) $y = (1 - x^2)^{10}$

   (b) $y = e^{\sqrt{x}}$

   (c) $f(x) = (x^4 + 7x - 4)^5$

   (d) $f(x) = \sqrt[4]{1 + 2x + x^3}$

   (e) $f(x) = xe^{-kx}$

   (f) $y = (2x^4 + 7)^5(8x - 4)^{-3}$

   (g) $y = e^{xcos\ x}$

   (h) $y = sin(tan\ 2x)$

   (i) $y = 2^{sin\ \pi x}$

   (j) $y = cos\sqrt{sin(tan\ \pi x)}$

   (k)
   $$y = \left(\frac{x^2 - 1}{x^2 + 1}\right)^3$$

   (l)
   $$y = \sqrt{\frac{x - 1}{x + 1}}$$

   (m)
   $$y = \frac{x}{\sqrt{x^2 + 1}}$$

5 Implicit Differentiation

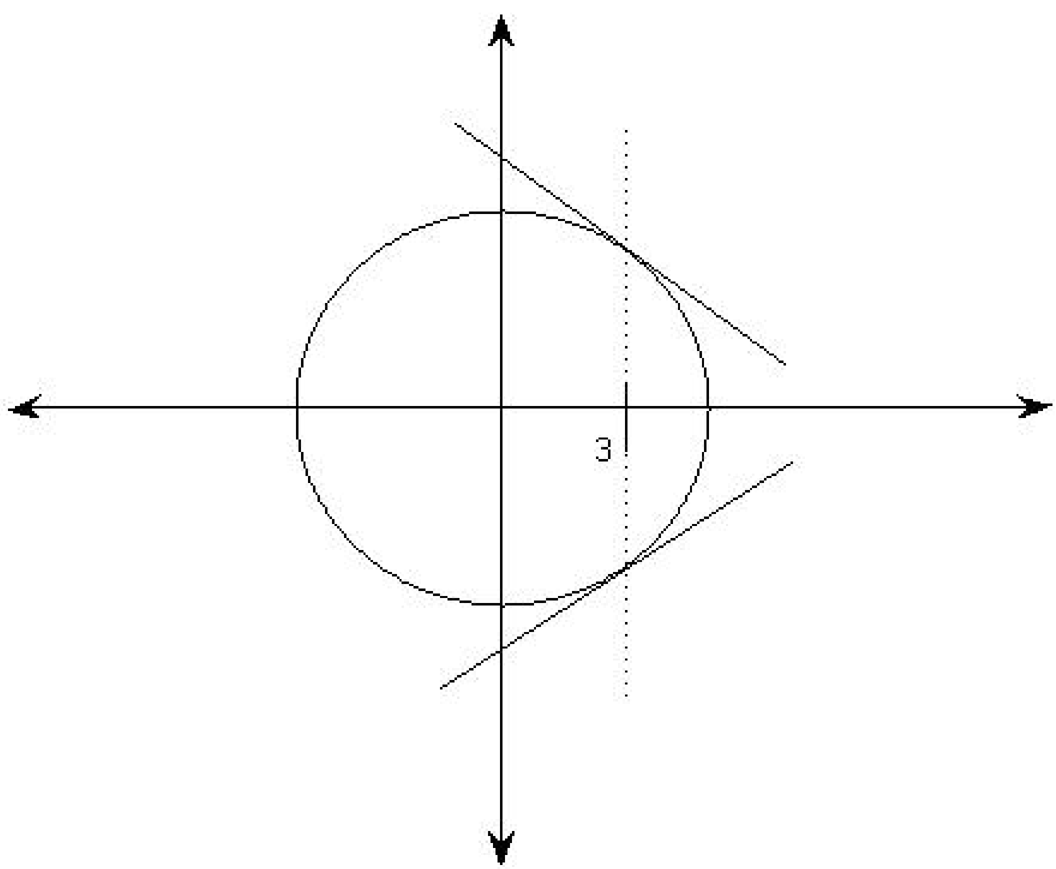
The functions that we have dealt with so far are *explicit functions*, that is, they can be written as y in terms of only stuff in x.

How would we deal with functions of the type $x^2 + y^2 = 25$?

If we attempt to solve this for y we would eventually get $y = \pm\sqrt{25 - x^2}$

In other words there are two possible equations here, which one should we use?

If I asked for the derivative at $x = 3$, which slope would I want?



We will need another method called IMPLICIT DIFFERENTIATION

The idea is to treat y as a function of x. Whenever you have **only** x, take the derivative as usual. When you have y, take the derivative and then **multiply** by $\frac{dy}{dx}$ *or* y' since y is a function of x.

ex 16 Find

$$\frac{d}{dx}\left[x^2 + y^2 = 25\right]$$

$$= \frac{d}{dx}(x^2)+\frac{d}{dx}(y^2) = \frac{d}{dx}(25) \implies 2x+2y\frac{dy}{dx} = 0 \implies \frac{dy}{dx} = -\frac{x}{y}$$

As you can clearly see the derivative depends on BOTH x AND y

ex 17 Find an equation of the tangent line to $x^3 + y^3 = 6xy$ at $(3,3)$

Be careful here since you will need the product rule on the right hand side. Why?

First the slope, y'

$$\frac{d}{dx}(x^3) + \frac{d}{dx}(y^3) = \frac{d}{dx}(6xy) \iff 3x^2 + 3y^2y' = 6\frac{d}{dx}(xy)$$

$$\iff 3x^2+3y^2y' = 6[x\frac{d}{dx}(y)+y\frac{d}{dx}(x)] \iff 3x^2+3y^2y' = 6xy'+6y$$

$$\iff 3y^2y' - 6xy' = 6y - 3x^2 \iff y'(3y^2 - 6x) = 6y - 3x^2$$

$$\iff y' = \frac{6y - 3x^2}{3y^2 - 6x}$$

As you can see the slope depends on x and y At $(3,3)$ we get

$$y' = \frac{6(3) - 3(3)^2}{3(3)^2 - 6(3)} = -1$$

So the equation is $y - 3 = -(x - 3)$

We can use Implicit Differentiation (ID) to find the derivatives of the inverse trig functions.

Recall:

$$sin^{-1}\ x = y \iff sin\ y = x \quad in \quad -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$$

Using ID on $sin\ y = x$ we get

$$\frac{d}{dx}[sin\ y = x] \iff cos\ y(y') = 1 \iff y' = \frac{1}{cos\ y}$$

However

$$cos\ y = \sqrt{1 - sin^2\ y} = \sqrt{1 - x^2} \ \ since \ \ sin\ y = x$$

So

$$\frac{d}{dx}(sin^{-1}\ x) = \frac{1}{\sqrt{1 - x^2}}$$

Also

$$\frac{d}{dx}(cos^{-1}\ x) = -\frac{1}{\sqrt{1 - x^2}}$$

and

$$\frac{d}{dx}(tan^{-1}\ x) = \frac{1}{1 + x^2}$$

Worksheet for Section 5

1. Find dy/dx by implicit differentiation:

   (a) $\sqrt{x+y} = 1 + x^2y^2$

   (b) $y^5 + x^2y^3 = 1 + ye^{x^2}$

2. Differentiate $y = cos^{-1}(e^{2x})$

Homework for Section 5

1. Find dy/dx by implicit differentiation:

   (a) $x^3 + y^3 = 1$

   (b) $x^2 + xy - y^2 = 4$

   (c) $x^4(x + y) = y^2(3x - y)$

   (d) $x^2y^2 + x\ sin\ y = 4$

   (e) $e^{x/y} = x - y$

   (f) $e^y\ cos\ x = 1 + sin\ xy$

2. Find the derivative and simplify:

   (a) $y = \sqrt{tan^{-1}\ x}$

   (b) $y = tan^{-1}(x - \sqrt{1 + x^2})$